The Lefschetz Fixed Point Theorem

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Overview

- Homology and cohomology
- Intersection of cycles
- The Lefschetz Fixed Point Theorem
- A good $p$-adic cohomology for the affine line
- Monsky-Washnitzer cohomology
Homology

• Chain complex $K$ is a sequence $\{C_n, \partial_n\}_{n \in \mathbb{Z}}$ of Abelian groups

\[
\cdots \leftarrow C_{n-1} \xleftarrow{\partial_n} C_n \xrightarrow{\partial_{n+1}} C_{n+1} \xleftarrow{\partial_{n+2}} \cdots
\]

and boundary maps (homomorphisms) such that $\partial_n \partial_{n+1} = 0$.

• Since $\partial_n \partial_{n+1} = 0$ one has $\text{Im} \partial_{n+1} \subset \text{Ker} \partial_n$ and

\[
H_n(K) := \text{Ker} \partial_n / \text{Im} \partial_{n+1}
\]

is the $n$-th homology group of $K$.

• Example: singular homology.
**Singular Homology**

- **$n$-simplex**: convex hull of $n + 1$ points $x_0, \ldots, x_n$ not in $n - 1$-dimensional subspace.

- **Standard $n$-simplex** $\sigma_n$: $x_0 = (1, 0, \ldots, 0), \ldots, x_n = (0, 0, \ldots, 1)$.

- A **singular $n$-simplex** of a topological space $X$ is a continuous function $\phi : \sigma_n \to X$.

- For each $0 \leq i \leq n$ we obtain a singular $n - 1$-simplex

  $$(\partial^{(i)} \phi)(t_0, \ldots, t_{n-1}) = \phi(t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$$

- **Boundary operator** $\partial$ is given by

  $$\partial_n = \partial^{(0)} - \partial^{(1)} + \cdots + (-1)^n \partial^{(n)}$$
**Singular Homology**

- Let $S_n(X)$ be free abelian group with basis singular $n$-simplices
  \[ S_n(X) = \left\{ \sum_{\phi} n_\phi \cdot \phi \mid n_\phi \neq 0 \text{ finitely many} \right\} \]

- By linearity $\partial_n : S_n(X) \leftarrow S_{n-1}(X)$ and $\partial_n \circ \partial_{n+1} = 0$.

- Element $c \in S_n(X)$ is $n$-cycle if $\partial_n(c) = 0$.

- Element $d \in S_n(X)$ is $n$-boundary if $d = \partial(e)$ for $e \in S_{n+1}(X)$.

- $n$-th singular homology group
  \[ H_n(K) := \text{Ker } \partial_n / \text{Im } \partial_{n+1} \]
Singular Homology
Cohomology

- **Cochain complex** is a sequence $\{C^n, d_n\}_{n \in \mathbb{Z}}$ of Abelian groups

  \[ \cdots \xrightarrow{d_{n-2}} C_{n-1} \xrightarrow{d_{n-1}} C_n \xrightarrow{d_n} C_{n+1} \xrightarrow{d_{n+1}} \cdots \]

  and **coboundary maps or differentials** such that $d_n d_{n-1} = 0$.

- Since $d_n d_{n-1} = 0$ one has $\text{Im } d_{n-1} \subset \text{Ker } d_n$ and

  \[ H^n(K) := \text{Ker } d_n / \text{Im } d_{n-1} \]

  is the $n$-th cohomology group of $K$.

- **Example**: algebraic de Rham cohomology.
Algebraic de Rham Cohomology

- \( X \) smooth, affine variety over \( K \) of char 0 with coordinate ring
  \[
  A := K[x_1, \ldots, x_n]/(f_1, \ldots, f_m)
  \]

- Module of Kähler differentials \( \Omega^1_{A/K} \) generated by \( dg \) with \( g \in A \)
  \[
  \Omega^1_{A/K} = (A \, dx_1 + \cdots + A \, dx_n)/(\sum_{i=1}^{m} A(\partial f_i/\partial x_1 \, dx_1 + \cdots + \partial f_i/\partial x_n \, dx_n)).
  \]

- \( \Omega^i_{A/K} = \wedge^i \Omega^1_{A/K} \) and \( d_i : \Omega^i_{A/K} \to \Omega^{i+1}_{A/K} \) exterior diff.

- Since \( d_{i+1} \circ d_i = 0 \) we get the de Rham complex \( \Omega_{A/K} \)
  \[
  0 \longrightarrow A \xrightarrow{d_0} \Omega^1_{A/K} \xrightarrow{d_1} \Omega^2_{A/K} \xrightarrow{d_2} \Omega^3_{A/K} \cdots
  \]
• $i$-th de Rham cohomology group of is defined as

$$H_{DR}^i(A/K) := \text{Ker } d_i / \text{Im } d_{i-1}$$
Intersection of Cycles
**Intersection of Cycles**

- Let $A$ and $B$ two cycles that intersect tranversely at point $p$.
- The **intersection number** of $A$ and $B$ is
  \[
  #(A \cdot B) = \sum_{p \in A \cap B} \iota_p(A \cdot B)
  \]
  - Intersection index $\iota_p(A \cdot B) \in \{-1, +1\}$ depends on orientation.
  - $#(A \cdot B)$ only depends on homology classes of $A$ and $B$!
- General: intersection number defines pairing
  \[
  H_k(M, \mathbb{Z}) \times H_{n-k}(M, \mathbb{Z}) \rightarrow \mathbb{Z}
  \]
- **Poincaré**: for any $k$-cycle $A$ on $M$ there is closed $(n-k)$-form $\varphi_A$
  \[
  #(A \cdot B) = \int_B \varphi_A
  \]
The Lefschetz Fixed Point Theorem

• Let $M$ be compact oriented manifold of dimension $n$ and $f : M \to M$ an endomorphism.

• The Lefschetz number of $f$ is defined as

$$L(f) = \sum_{i=0}^{n} (-1)^i \text{Trace}(f_*|H^{i}_{DR}(M)).$$

• A point $p \in M$ is called a fixed point of $f$ if

$$f(p) = p$$

• Question: what is $\#\{p \in M \mid f(p) = p\}$?
The Lefschetz Fixed Point Theorem

- Diagonal $\Delta \subset M \times M$ and graph $\Gamma_f = \{(p, f(p))|p \in M\}$ of $f$.

**fixed point = intersection of $\Delta$ and $\Gamma_f$**
The Lefschetz Fixed Point Theorem

• If $f$ has only nondegenerate fixed points then

$$\#(\Delta \cdot \Gamma_f)_{M \times M} = \sum_{f(p) = p} \iota_f(p)$$

• The Lefschetz Fixed Point Formula

$$\sum_{f(p) = p} \iota_f(p) = L(f) = \sum_i (-1)^i \text{Trace}(f_*|H^i_{DR}(M))$$

• Proof:

$$\#(\Delta \cdot \Gamma_f)_{M \times M} = \int_{\Gamma_f} \varphi_\Delta$$

• $\varphi_\Delta$ Poincaré dual of homology class of diagonal.
The Lefschetz Fixed Point Theorem

• Corollary 1: \( \# \{ p \in M : f(p) = p \} \geq |L(f)| \).

• Corollary 2: If \( L(f) \neq 0 \), then \( f \) has a fixed point.

• Theorem: for analytic cycles \( V \) and \( W \) of compact complex manifold meeting transversally \( \nu_p(V \cdot W) = +1 \).

• Lefschetz Fixed Point Theorem: Let \( M \) be a compact complex analytic manifold and \( f : M \to M \) an analytic map. Assume that \( f \) only has isolated nondegenerate fixed points then

\[
\# \{ p \in M \mid f(p) = p \} = L(f) = \sum_i (-1)^i \text{Trace}(f_*|H^i_{DR}(M))
\]
A $p$-adic Cohomology of the Affine Line

- Frobenius $\overline{F}: \overline{\mathbb{F}}_p \rightarrow \overline{\mathbb{F}}_p : x \mapsto x^p$ then $x \in \mathbb{F}_p$ iff $\overline{F}(x) = x$.

- Consider $\overline{C}: xy - 1 = 0$ with coordinate ring $\overline{\mathbb{A}} = \mathbb{F}_p[x, 1/x]$, then
  
  $$N_r = \# \overline{C}(\mathbb{F}_{p^r}) = \# \text{ fixed points of } \overline{F}^r = p^r - 1$$

- Construct de Rham cohomology in characteristic $p$?
  
  - Only possible to compute $N_r \pmod{p}$.
  - $\Omega^1(\overline{\mathbb{A}}) := \overline{\mathbb{A}} dx/(d \overline{\mathbb{A}})$ is infinite dimensional.
  - $x^k \, dx$ with $k \equiv -1 \pmod{p}$ cannot be integrated.
\textbf{\(p\)-ADIC NUMBERS}

- \(p\)-adic norm \(| \cdot |_p\) of \(r \neq 0 \in \mathbb{Q}\) is
  \[
  |r|_p = p^{-\rho}, \quad r = p^{\rho} u / v, \quad \rho, u, v \in \mathbb{Z}, \quad p \nmid u, p \nmid v.
  \]

- Field of \(p\)-adic numbers \(\mathbb{Q}_p\) is completion of \(\mathbb{Q}\) w.r.t. \(| \cdot |_p\),
  \[
  \sum_{m} a_i p^i, \quad a_i \in \{0, 1, \ldots, p - 1\}, \quad m \in \mathbb{Z}.
  \]

- \(p\)-adic integers \(\mathbb{Z}_p\) is the ring with \(| \cdot |_p \leq 1\) or \(m \geq 0\).

- Unique maximal ideal \(M = \{x \in \mathbb{Q}_p \mid |x|_p < 1\} = p\mathbb{Z}_p\) and \(\mathbb{Z}_p / M \cong \mathbb{F}_p\).
A $p$-adic Cohomology of the Affine Line

First attempt: lift situation to $\mathbb{Z}_p$ and try again?

- Consider two lifts to $\mathbb{Z}_p$

$$A_1 = \mathbb{Z}_p[x, 1/x] \quad \text{and} \quad A_2 = \mathbb{Z}_p[x, 1/(x(1+px))]$$

- $A_1$ and $A_2$ are not isomorphic; both $x$ and $1+px$ invertible in $A_2$.

- $H^1_{DR}(A_1/\mathbb{Q}_p) = \langle \frac{dx}{x} \rangle$ and $H^1_{DR}(A_2/\mathbb{Q}_p) = \langle \frac{dx}{x}, \frac{dx}{1+px} \rangle$.

- Frobenius does not always lift:

  - Example: $\overline{A} = \mathbb{F}_3[x]/(x^2 - 2)$ and $A = \mathbb{Z}_3[x]/(x^2 - 2)$
A $p$-adic Cohomology of the Affine Line

Second attempt: use $p$-adic completion.

$$A_1^\infty \cong A_2^\infty \cong \{ \sum_{i \in \mathbb{Z}} \alpha_i x^i \in \mathbb{Z}_p[[x, 1/x]] \mid \lim_{|i| \to +\infty} \alpha_i = 0 \}$$

- However: $H_{DR}^1(A^\infty / \mathbb{Q}_p)$ is again infinite dimensional!
- $\sum_i p^i x^{p^{i-1}}$ is in $A^\infty$ but integral $\sum_i x^{p^i}$ is not.
- Convergence property lost in integration.
A $p$-adic Cohomology of the Affine Line

Third attempt: consider the dagger ring or weak completion

$$A^\dagger = \left\{ \sum_{i \in \mathbb{Z}} \alpha_i x^i \in \mathbb{Z}_p[[x, 1/x]] \mid \exists \epsilon \in \mathbb{R}_{>0}, \delta \in \mathbb{R} : v_p(\alpha_i) \geq \epsilon |i| + \delta \right\}$$

- Note: $A_1^\dagger$ is isomorphic to $A_2^\dagger$, since $1 + px$ invertible in $A_1^\dagger$.

$$\frac{1}{1 + px} = \sum_{i=0}^{\infty} (-1)^i p^i x^i$$
A $p$-adic Cohomology of the Affine Line

- Monsky-Washnitzer := de Rham cohomology of $A^\dagger \otimes \mathbb{Q}_p$
- $H^1(\overline{A}/\mathbb{Q}_p) = (A^\dagger \otimes \mathbb{Q}_p)dx/(d(A^\dagger \otimes \mathbb{Q}_p))$ and clearly for $k \neq -1$
  \[ x^k dx = d\left(\frac{x^{k+1}}{k+1}\right) \]

- Conclusion: $H^1(\overline{A}/\mathbb{Q}_p)$ has basis $\frac{dx}{x}$
- Lifting Frobenius $F$ to $A^\dagger$: infinitely many possibilities
  \[ F(x) \in x^p + pA^\dagger \]
- Examples: $F_1(x) = x^p$ or $F_2(x) = x^p + p$
A $p$-adic Cohomology of the Affine Line

• Action of $F_1$ on basis $\frac{dx}{x}$ is given by

$$F_1^* \left( \frac{dx}{x} \right) = \frac{d(F_1(x))}{F_1(x)} = \frac{d(x^p)}{x^p} = p \frac{dx}{x}$$

• Action of $F_2$ on basis $\frac{dx}{x}$ is given by

$$F_2^* \left( \frac{dx}{x} \right) = \frac{d(F_2(x))}{F_2(x)} = \frac{d(x^p + p)}{x^p + p} = p \frac{x^{p-1}}{x^p + p} \frac{dx}{x} = \frac{p}{1 + px^{-p}} \frac{dx}{x}$$

• Power series expansion: $(1 + px^{-p})^{-1} = \sum_{i=0}^{\infty} (-1)^i p^i x^{-ip} \in A^+$

$$F_2^* \left( \frac{dx}{x} \right) = p \frac{dx}{x} + d \left( \sum_{i=1}^{\infty} \left( -1 \right)^{i+1} \frac{p^{i-1}}{i} x^{-ip} \right)$$
**A p-adic Cohomology of the Affine Line**

- Action of $F_1$ and $F_2$ are equal on $H^1(\overline{A}/\mathbb{Q}_p)$!

$$F_*(\frac{dx}{x}) = p \frac{dx}{x} \Rightarrow F_*^{-1}\left(\frac{dx}{x}\right) = \frac{1}{p} \frac{dx}{x}$$

- Lefschetz Trace formula applied to $\overline{C}$ gives

$$\#\overline{C}(\mathbb{F}_{p^r}) = \text{Trace}\left((pF_*^{-1})^r|H^0(\overline{C}/\mathbb{Q}_p)\right) - \text{Trace}\left((pF_*^{-1})^r|H^1(\overline{C}/\mathbb{Q}_p)\right)$$

- Conclusion:

$$\#\overline{C}(\mathbb{F}_{p^r}) = p^r - 1$$
Monsky-Washnitzer cohomology

- $\overline{X}$ smooth affine variety over $\mathbb{F}_q$ with coordinate ring $\overline{A}$.
- Exists $A := \mathbb{Z}_q[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ with $A \otimes_{\mathbb{Z}_q} \mathbb{F}_q \cong \overline{A}$.
- Dagger ring or weak completion $A^\dagger$ is defined

$$A^\dagger := \mathbb{Z}_q \langle x_1, \ldots, x_n \rangle^\dagger / (f_1, \ldots, f_m)$$

with $\mathbb{Z}_q \langle x_1, \ldots, x_n \rangle^\dagger$ overconvergent power series

$$\left\{ \sum_I a_I x^I \in \mathbb{Z}_q[[x_1, \ldots, x_n]] \mid \liminf_{|I| \to \infty} \frac{v_p(\alpha_I)}{|I|} > 0 \right\}$$

- M-W cohomology is the de Rham cohomology of $A^\dagger \otimes \mathbb{Q}_q$. 
Monsky-Washnitzer cohomology

• Definition only depends on $\overline{A}$ and not on choices made!
• Every morphism $\overline{G} : \overline{A} \to \overline{B}$ lifts to $G : A^\dagger \to B^\dagger$.
• Induced map on $H^i(\overline{A}/\mathbb{Q}_q) \to H^i(\overline{B}/\mathbb{Q}_q)$ only depends on $\overline{G}$.
• Cohomology groups $H^i(\overline{A}/\mathbb{Q}_q)$ are finite dimensional.
• Lefschetz trace formula: for $X$ of dimension $d$

$$N_r = \sum_{i=0}^{d} (-1)^i \text{Tr} \left((q^d F_*^{-1})^r|H^i(\overline{X}/\mathbb{Q}_q)\right)$$

• Let $\overline{C}$ be a projective, smooth curve of genus $g$ over $\mathbb{F}_q$
  - $S$ a set of $m$ $\mathbb{F}_q$-points and $\overline{A}$ coordinate ring of $\overline{C} \setminus S$
  
  $$\dim H^1(\overline{A}/\mathbb{Q}_q) = 2g + m - 1$$