Irredundant lattice piecewise affine representations and their applications in explicit model predictive control

Jun Xu

China University of Petroleum (Beijing)
Delft University of Technology

July 14, 2014

1. Introduction
2. Full lattice PWA representation
3. Irredundant lattice PWA representation
   - Implicant and prime implicant
   - Necessary and Sufficient condition for irredundancy
   - Removing redundant literals
4. Application in explicit MPC
   - Explicit linear MPC
   - Simulation
5. Conclusions and future work

Lattice piecewise affine representation

- Lattice piecewise affine (PWA) representation is proposed to generate PWA functions for the approximation of a nonlinear function and is able to represent any continuous PWA function.
- The lattice PWA representation can be described as

$$f = \min_{i=1,\ldots,N_1} \{\max_{j\in\bar{I}_i}\{\ell_j\}\} \qquad \text{conjunctive form}$$

or

$$f = \max_{i=1,\ldots,N_2} \{\min_{j\in\tilde{I}_i}\{\ell_j\}\} \qquad \text{disjunctive form}$$

Explicit linear model predictive control

- The optimal controller of linear model predictive control (MPC) is continuous PWA with respect to the state x_t.
- The optimal solution is given by the MPT toolbox in the form of subregions plus local affine functions defined on them.
- The storage requirement & the online complexity.

Definition of continuous PWA function:

domain partition $\mathbb{D} = \cup_i^N \Omega_i$

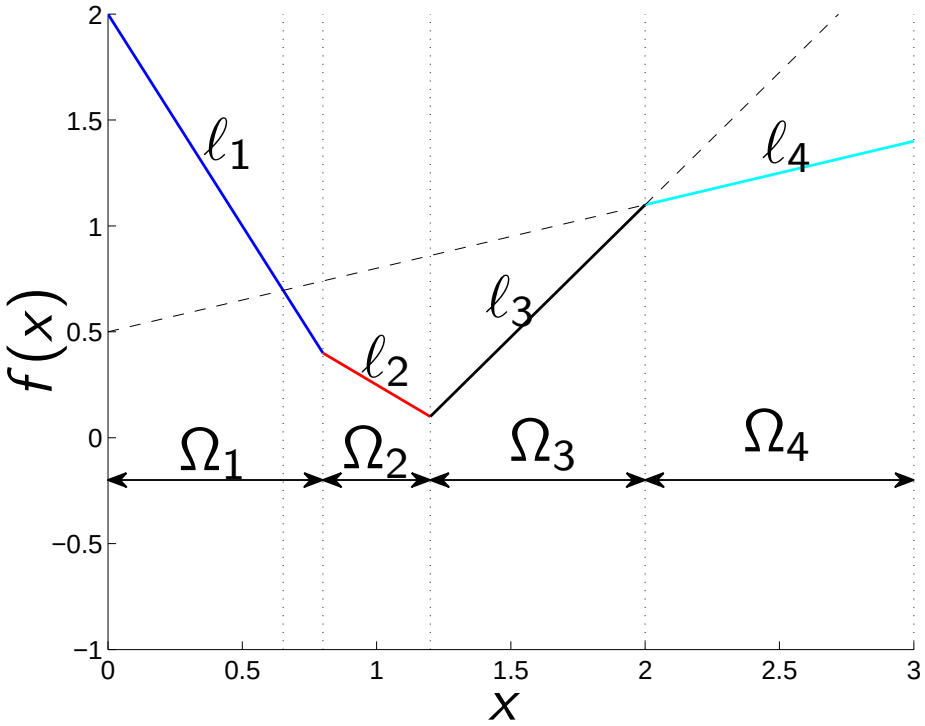
local affine $f \equiv \ell_{\mathrm{loc}(i)}$ in Ω_i.

continuity $\ell_{\mathrm{loc}(i)}(x) = \ell_{\mathrm{loc}(j)}(x),\ \forall x \in \Omega_i \cap \Omega_j,\ \Omega_i \cap \Omega_j \neq \emptyset.$

Full lattice PWA representation

$$f = \max_{i=1,\ldots,N} \{ \min_{j \in K_{\geq,i}} \{\ell_j(x)\}\}, \forall x \in \mathbb{D}$$

in which $K_{\geq,i} = \{j | \ell_j(x) \geq \ell_{\mathrm{loc}(i)}(x), \forall x \in \Omega_i\}$.



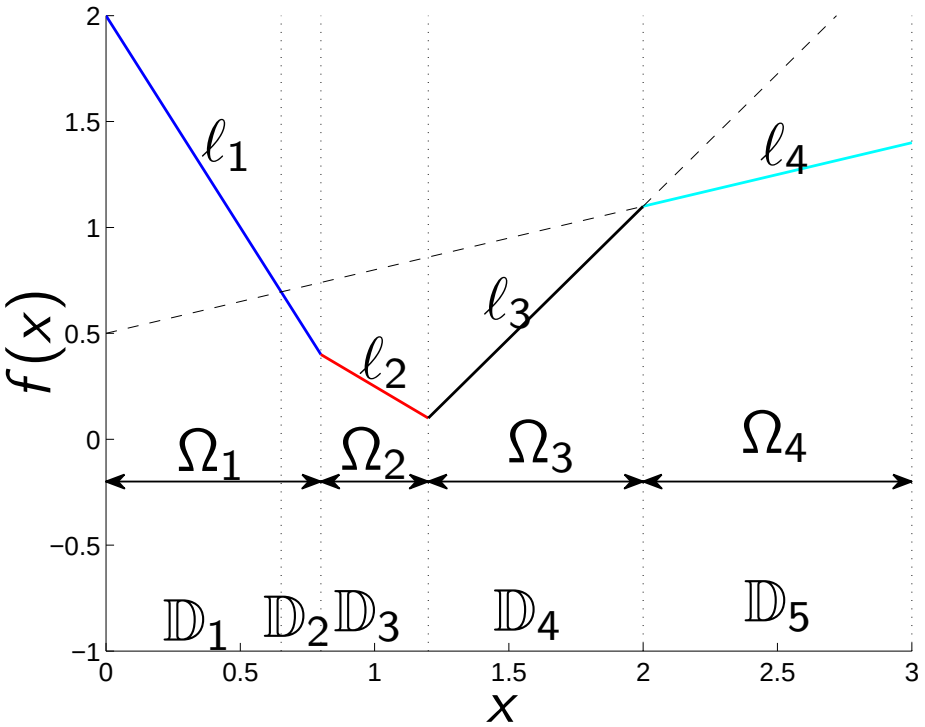
$$K_{\geq,1} = \{1\}, K_{\geq,2} = \{2,4\}$$
$$K_{\geq,3} = \{3,4\}, K_{\geq,4} = \{4,3\}$$
$$f = \max\{\ell_1, \min\{\ell_2, \ell_4\}, \min\{\ell_3, \ell_4\}, \min\{\ell_3, \ell_4\}\}$$

In $[0, 3]$, f equals,

$$\max\{\ell_1, \ell_2, \min\{\ell_3, \ell_4\}\}$$

f is Redundant!

Base region



Subregions: $\Omega_1, \Omega_2, \Omega_3, \Omega_4$
Base regions: $\mathbb{D}_1, \ldots, \mathbb{D}_5$

$\Omega_1 = \mathbb{D}_1 \cup \mathbb{D}_2$,
$\Omega_2 = \mathbb{D}_3, \Omega_3 = \mathbb{D}_4$, and
$\Omega_4 = \mathbb{D}_5$.

Implicant (Analog to **Boolean algebra**)

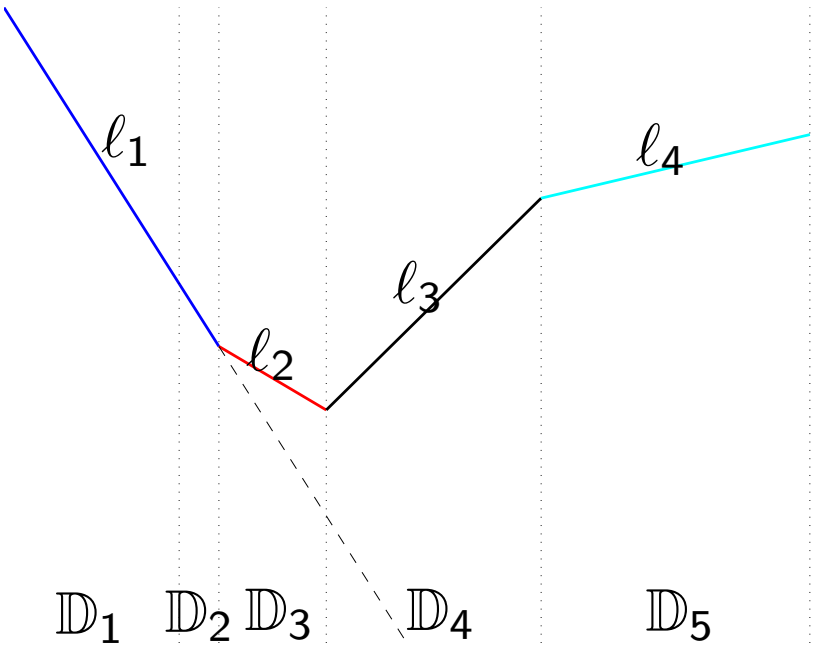
The term $T_i = \min_{j \in I_i} \{\ell_j\}$ is an ***implicant*** of continuous PWA function f if

1. $T_i \leq f$ in $\mathbb{D}$.
2. $\exists$ base region $\mathbb{D}_k$ such that $T_i \equiv f$ in $\mathbb{D}_k$.

Prime implicant: $T_i = \min_{j \in I_i} \{\ell_j\}$ is a ***prime implicant*** of continuous PWA function f if

1. T_i is an implicant of f.
2. no literal ℓ_j can be deleted from T_i such that the remaining term is still an implicant of f.

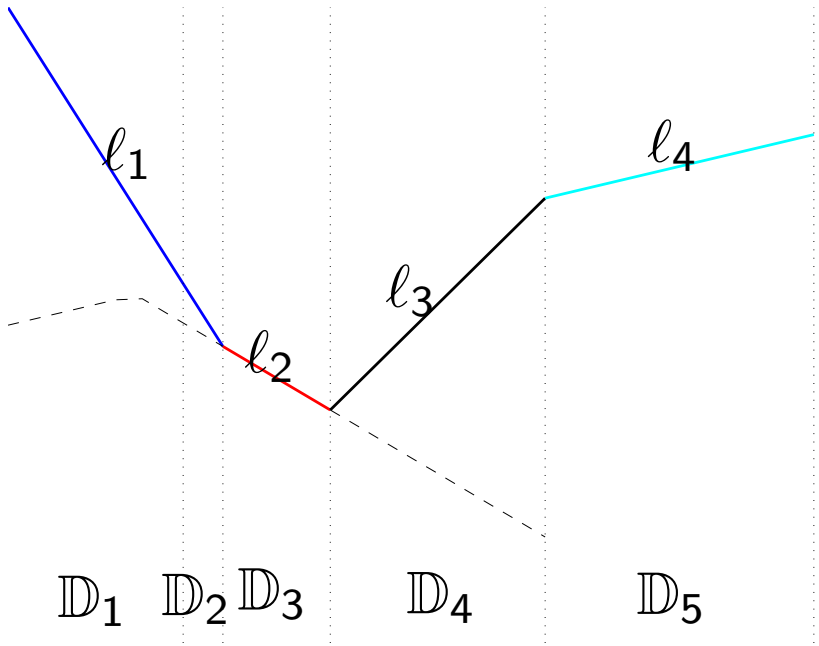
Implicant and prime implicant



Implicant:

$$T_1^{\mathrm{F}} = \ell_1$$

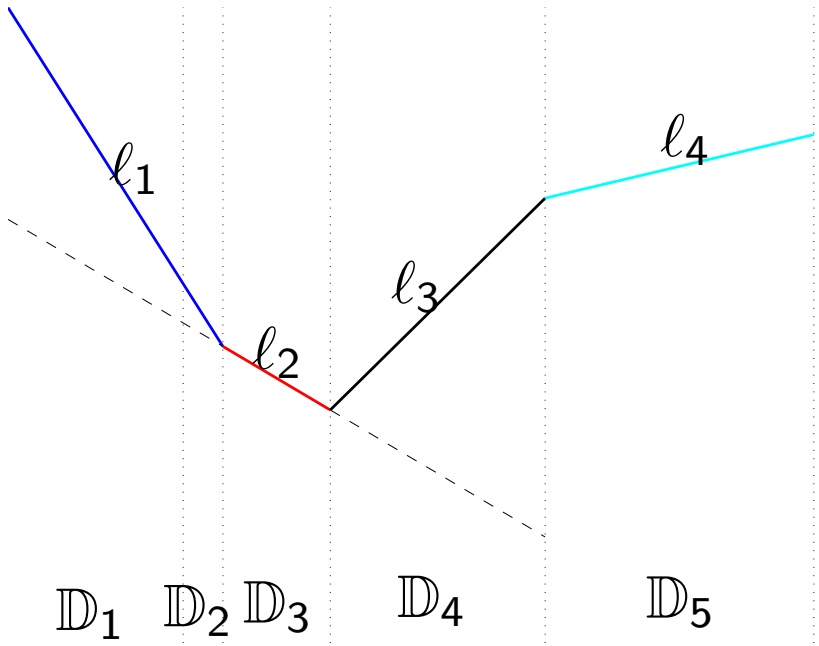
is a Prime implicant



Implicant:

$$T_2^{\mathrm{F}} = \min\{\ell_2, \ell_4\}$$

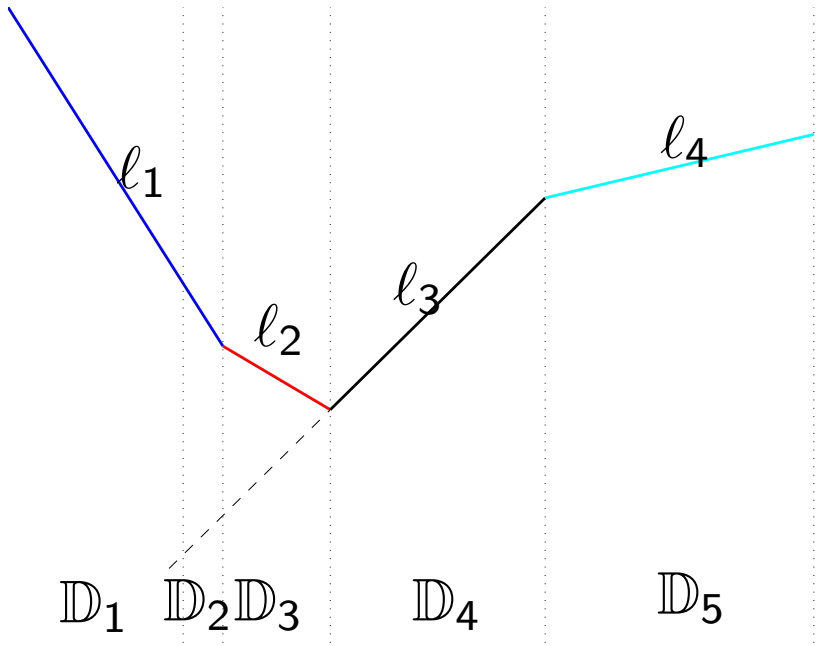
is not a prime implicant.



Implicant:

$$T_2 = \ell_2$$

is a Prime implicant



Implicant:

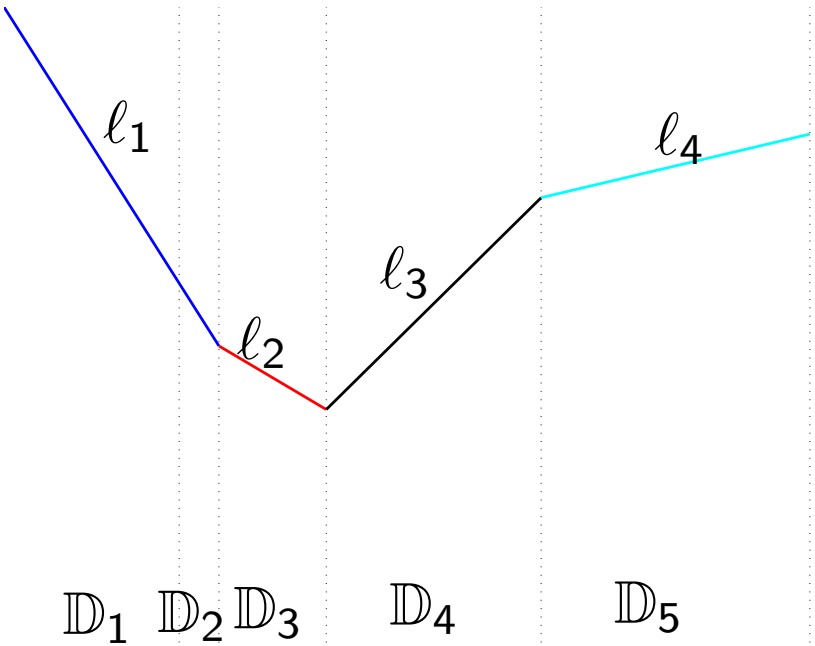
$$T_3^{\mathrm{F}} = \min\{\ell_3, \ell_4\}$$

is a Prime implicant

Each term in ther full lattice PWA representation is an implicant of f.

Active in the base region

The implicant $T_i = \min_{j \in I_i}\{\ell_j\}$ is called *active* in the base region $\mathbb{D}_k$, if $T_i \equiv f$ in $\mathbb{D}_k$. The indices of all base regions in which T_i is active constitute an index set $\mathcal{A}(T_i)$.



$$\mathcal{A}(\ell_1) = \{1, 2\},$$
$$\mathcal{A}(\min\{\ell_2, \ell_4\}) = \{3\},$$
$$\mathcal{A}(\min\{\ell_3, \ell_4\}) = \{4, 5\},$$
$$\mathcal{A}(\ell_2) = \{3\}$$

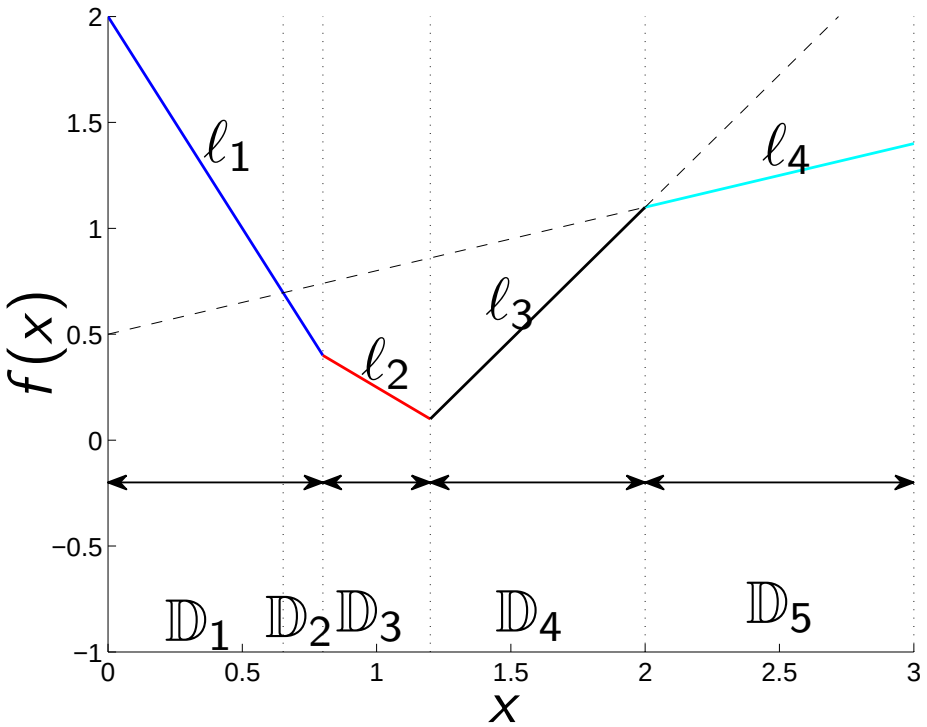
Theorem

A lattice PWA representation

$$f_{\mathrm{L}} = \max_{i=1,\ldots,\tilde{N}} \{\min_{j\in I_i}\{\ell_j\}\}$$

satisfying $f_{\mathrm{L}} = f$ *in* $\mathbb{D}$ *is irredundant if and only if:*

1. *Each term* $T_i = \min_{j\in I_i}\{\ell_j\}$ *is a prime implicant of* f.
2. $\cup_{i=1}^{\tilde{N}} \mathcal{A}(T_i) = \{1,\ldots,N\}$, *in which* N *is the number of base regions.*
3. $\mathcal{A}(T_j) \not\subseteq \cup_{i=1,i\neq j}^{\tilde{N}} \mathcal{A}(T_i)$, $\forall j \in \{1,\ldots,\tilde{N}\}$.



$$f = \max\{\ell_1, \min\{\ell_2, \ell_4\}, \min\{\ell_3, \ell_4\}, \min\{\ell_3, \ell_4\}\}$$

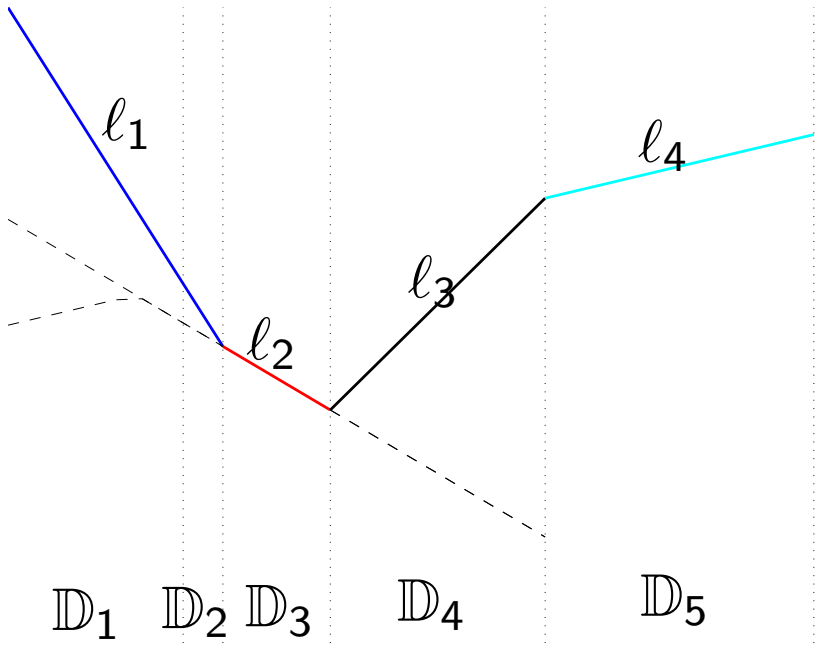
is redundant.

$f = \max\{\ell_1, \ell_2, \min\{\ell_3, \ell_4\}\}$ is irredundant.

Theorem

Given a term $T_i^{\mathrm{F}} = \min\limits_{j \in K_{\geq,i}} \{\ell_j\}$ *in the full lattice PWA representation. The term* $T_i = \min\limits_{j \in I_i} \{\ell_j\}$ *with* $I_i \subsetneq K_{\geq,i}$ *is an implicant of* f *if and only if* $\forall t \in K_{\geq,i} \setminus I_i$, $\forall k \in \mathcal{A}(\ell_t)$, *we have*

$$T_i(x) \leq \ell_t(x), \forall x \in \mathbb{D}_k.$$



Implicants:

$$T_2^{\mathrm{F}} = \min\{\ell_2, \ell_4\}$$
$$T_2 = \ell_2$$

$\mathcal{A}(\ell_4) = \{5\}$, and $\ell_2 \leq \ell_4$ in $\mathbb{D}_5$.

Constrained regulation problem solved by MPC at time step t:

$$\begin{aligned}
\min_{U} \quad & \left\{ J(U, x_t) = x_{t+N_y}^T P x_{t+N_y} + \sum_{k=0}^{N_y-1} \left[x_{t+k}^T Q x_{t+k} + u_{t+k}{}^T R u_{t+k} \right] \right\} \\
s.t. \quad & y_{\min} \leq y_{t+k} \leq y_{\max}, k = 1, \ldots, N_y, \\
& u_{\min} \leq u_{t+k} \leq u_{\max}, k = 0, 1, \ldots, N_y - 1, \\
& x_{t+k+1} = A x_{t+k} + B u_{t+k}, k = 0, 1, \ldots, N_y - 1, \\
& y_{t+k} = C x_{t+k}, k = 1, \ldots, N_y, \\
& u_{t+k} = K x_{t+k}, k = N_u, \ldots, N_y - 1,
\end{aligned}$$

The control law $u_t = f(x_t)$, $f : \mathbb{D} \to \mathbb{R}^m$ defined by the optimization problem is a *continuous and PWA* function.
Explicit: online optimization $\longrightarrow$ offline parametric optimization + online function evaluation.

Example

A discrete-time double integrator (x_t is 2-dimensional).
The domain partition of the explicit MPC controller:

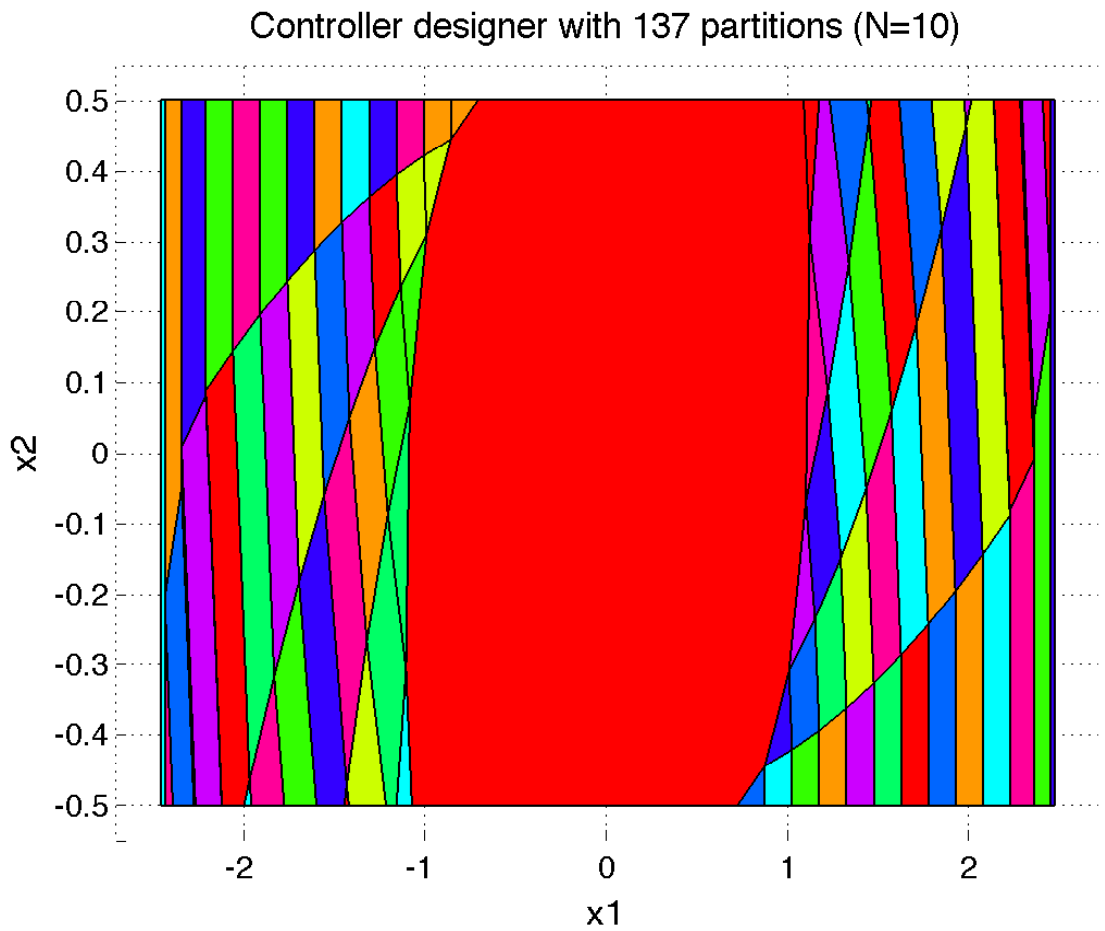


Table : Performance of irredundant lattice PWA representations with respect to different prediction horizon.

$N_y = N_u$	N_{region}	$N_{\text{para,pwa}}$	N_{affine}	N_{terms}	$N_{\text{para,LR}}$
2	31	477	17	9	80
6	111	1677	29	13	148
10	137	2073	27	11	133
14	157	2385	27	11	135
18	155	2349	27	11	135
20	155	2349	27	11	135

Conclusions:

1. The irredundant lattice PWA representation has been proposed, which can represent all continuous PWA function.
2. Necessary and sufficient conditions for irredundancy have been proposed.
3. Necessary and sufficient condition for replacing an implicant with simpler implicants.
4. A worked example is provided to express the explicit solution of a linear MPC problem.

Future work:

- Derive the minimal max-min form for a general continuous PWA function.

Thank you! ☺